Chapter 2

2.1 1-Forms

One of the most puzzling ideas in elementary calculus is the idea of the differential. In the usual definition, the differential of a dependent variable $y = f(x)$, is given in terms of the differential of the independent variable by $dy = f'(x)dx$. The problem is with the quantity dx. What does dx mean? What is the difference between Δx and dx? How much "smaller" than Δx does dx have to be? There is no trivial resolution to this question. Most introductory calculus texts evade the issue by treating dx as an arbitrarily small quantity (which lacks mathematical rigor) or by simply referring to dx as an infinitesimal (a term introduced by Newton for an idea that could not otherwise be clearly defined at the time.)

In this section we introduce linear algebraic tools that will allow us to interpret the differential in terms of an linear operator.

2.1 Definition Let $\mathbf{p} \in \mathbf{R}^n$, and let $T_p(\mathbf{R}^n)$ be the tangent space at $\mathbf{p}$. A **1-form at p** is a linear map ϕ from $T_p(\mathbf{R}^n)$ into $\mathbf{R}$. We recall that such a map must satisfy the following properties

$$\begin{aligned} &\text{a)} \quad && \phi(X_p) \in \mathbf{R}, && \forall X_p \in \mathbf{R}^n \\ &\text{b)} \quad && \phi(aX_p + bY_p) = a\phi(X_p) + b\phi(Y_p), && \forall a, b \in \mathbf{R},\ X_p, Y_p \in T_p(\mathbf{R}^n) \end{aligned} \tag{2.1}$$

A **1-form** is a smooth choice of a linear map ϕ as above for each point in the space.

2.2 Definition Let $f : \mathbf{R}^n \to \mathbf{R}$ be a real-valued C^∞ function. We define the differential df of the function as the 1-form such that

$$df(X) = X(f) \tag{2.2}$$

for every vector field in X in $\mathbf{R}^n$.

In other words, at any point $\mathbf{p}$, the differential df of a function is an operator which assigns to a tangent vector X_p, the directional derivative of the function in the direction of that vector

$$df(X)(p) = X_p(f) = \quad f(p) \cdot \mathbf{X}(p) \tag{2.3}$$

In particular, if we apply the differential of the coordinate functions x^i to the basis vector fields, we get

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial x^i}{\partial x^j} = \delta^i_j \tag{2.4}$$

The set of all linear functionals on a vector space is called the **dual** of the vector space. It is an standard theorem in linear algebra that the dual of a vector space is also a vector space of the same dimension. Thus, the space $T^\star_p\mathbf{R}^n$ of all 1-forms at $\mathbf{p}$ is a vector space which is the dual of the tangent space $T_p\mathbf{R}^n$. The space $T^\star_p(\mathbf{R}^n)$ is called the **cotangent space** of $\mathbf{R}^n$ at the point $\mathbf{p}$. Equation (2.4) indicates that the set of differential forms $\{(dx^1)_p, \dots, (dx^n)_p\}$ constitutes the basis of the cotangent space which is dual to the standard basis $\{(\frac{\partial}{\partial x^1})_p, \dots (\frac{\partial}{\partial x^n})_p\}$ of the tangent space. The union of all the cotangent spaces as $\mathbf{p}$ ranges over all points in $\mathbf{R}^n$ is called the cotangent bundle $T^*(\mathbf{R}^n)$.

2.3 Proposition Let f be any smooth function in $\mathbf{R}^n$ and let $\{x^1, \dots x^n\}$ be coordinate functions in a neighborhood U of a point $\mathbf{p}$. Then, the differential df is given locally by the expression

$$
\begin{aligned}
df &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \qquad (2.5)\\
&= \frac{\partial f}{\partial x^i} dx^i
\end{aligned}
$$

Proof: The differential df is by definition a 1-form, so, at each point, it must be expressible as a linear combination of the basis elements $\{(dx^1)_p, \dots, (dx^n)_p\}$. Therefore, to prove the proposition, it suffices to show that the expression 2.5 applied to an arbitrary tangent vector, coincides with definition 2.2. To see this, consider a tangent vector $X_p = a^j(\frac{\partial}{\partial x^j})_p$ and apply the expression above

$$
\begin{aligned}
(\frac{\partial f}{\partial x^i} dx^i)_p(X_p) &= (\frac{\partial f}{\partial x^i} dx^i)(a^j \frac{\partial}{\partial x^j})(p) \qquad (2.6)\\
&= a^j (\frac{\partial f}{\partial x^i} dx^i)(\frac{\partial}{\partial x^j})(p)\\
&= a^j (\frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial x^j})(p)\\
&= a^j (\frac{\partial f}{\partial x^i} \delta^i_j)(p)\\
&= (\frac{\partial f}{\partial x^i} a^i)(p)\\
&= f(p) \cdot \mathbf{X}(p)\\
&= df(X)(p)
\end{aligned}
$$

The definition of differentials as linear functionals on the space of vector fields is much more satisfactory than the notion of infinitesimals, since the new definition is based on the rigorous machinery of linear algebra. If α is an arbitrary 1-form, then locally

$$
\alpha = a_1(\mathbf{x}) dx^1 + , \dots + a_n(\mathbf{x}) dx^n, \qquad (2.7)
$$

where the coefficients a_i are C^∞ functions. A 1-form is also called a **covariant tensor** of rank 1, or just simply a **covector**. The coefficients $(a_1, \dots, a_n)$ are called the **covariant** components of the covector. We will adopt the convention to always write the covariant components of a covector with the indices down. Physicists often refer to the covariant components of a 1-form as a covariant vector and this causes some confusion about the position of the indices. We emphasize that not all one forms are obtained by taking the differential of a function. If there exists a function f, such that $\alpha = df$, then the one form α is called **exact**. In vector calculus and elementary physics, exact forms are important in understanding the path independence of line integrals of conservative vector fields.

As we have already noted, the cotangent space $T^*_p(\mathbf{R}^n)$ of 1-forms at a point $\mathbf{p}$ has a natural vector space structure. We can easily extend the operations of addition and scalar multiplication to the space of all 1-forms by defining

$$\begin{aligned}(\alpha+\beta)(X) &= \alpha(X)+\beta(X) \\ (f\alpha)(X) &= f\alpha(X)\end{aligned} \tag{2.8}$$

for all vector fields X and all smooth functions f.

2.2 Tensors and Forms of Higher Rank

As we mentioned at the beginning of this chapter, the notion of the differential dx is not made precise in elementary treatments of calculus, so consequently, the differential of area $dxdy$ in $\mathbf{R}^2$, as well as the differential of surface area in $\mathbf{R}^3$ also need to be revisited in a more rigorous setting. For this purpose, we introduce a new type of multiplication between forms which not only captures the essence of differentials of area and volume, but also provides a rich algebraic and geometric structure which is vast generalization of cross products (which only make sense in $\mathbf{R}^3$) to Euclidean spaces of all dimensions.

2.4 Definition A map $\phi : T(\mathbf{R}^n)\times T(\mathbf{R}^n) \longrightarrow \mathbf{R}$ is called a **bilinear** map on the tangent space, if it is linear on each slot. That is

$$\begin{aligned}\phi(f^1X_1+f^2X_2,Y_1) &= f^1\phi(X_1,Y_1)+f^2\phi(X_2,Y_1) \\ \phi(X_1,f^1Y_1+f^2Y_2) &= f^1\phi(X_1,Y_1)+f^2\phi(X_1,Y_2), \qquad \forall X_i,Y_i\in T(\mathbf{R}^n),\;\; f^i\in C^\infty\mathbf{R}^n\end{aligned}$$

Tensor Products

2.5 Definition Let α and β be 1-forms. The **tensor product** of α and β is defined as the bilinear map $\alpha\otimes\beta$ such that

$$(\alpha\otimes\beta)(X,Y)=\alpha(X)\beta(Y) \tag{2.9}$$

for all vector fields X and Y.

Thus, for example, if $\alpha = a_i dx^i$ and $\beta = b_j dx^j$, then,

$$\begin{aligned}(\alpha\otimes\beta)(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}) &= \alpha(\frac{\partial}{\partial x^k})\beta(\frac{\partial}{\partial x^l}) \\ &= (a_i dx^i)(\frac{\partial}{\partial x^k})(b_j dx^j)(\frac{\partial}{\partial x^l}) \\ &= a_i\delta^i_k b_j \delta^j_l \\ &= a_k b_l\end{aligned}$$

A quantity of the form $T = T_{ij}dx^i\otimes dx^j$ is called a **covariant tensor of rank 2,** and we may think of the set $\{dx^i\otimes dx^j\}$ as a basis for all such tensors. We must caution the reader again that there is possible confusion about the location of the indices, since physicists often refer to the components $T_i j$ as a covariant tensor.

In a similar fashion, one can also define the tensor product of vectors X and Y as the bilinear map $X\otimes Y$ such that

$$(X\otimes Y)(f,g)=X(f)Y(g) \tag{2.10}$$

for any pair of arbitrary functions f and g.

If $X = a^i \frac{\partial}{\partial x^i}$ and $Y = b^j \frac{\partial}{\partial x^j}$, then, the components of $X \otimes Y$ in the basis $\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ are simply given by $a^i b^j$. Any bilinear map of the form

$$T = T^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \tag{2.11}$$

is called a contravariant tensor of rank 2 in $\mathbf{R}^n$.

The notion of tensor products can easily be generalized to higher rank, and in fact one can have tensors of mixed ranks. For example, a tensor of contravariant rank 2 and covariant rank 1 in $\mathbf{R}^n$ is represented in local coordinates by an expression of the form

$$T = T^{ij}{}_k \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes dx^k.$$

This object is also called a tensor of type $T^{2,1}$. Thus, we may think of a tensor of type $T^{2,1}$ as map with three input slots. The map expects two functions in the first two slots and a vector in the third one. The action of the map is bilinear on the two functions and linear on the vector. The output is a real number. An assignment of a tensor to each point in $\mathbf{R}^n$ is called a tensor field.

Inner Products

Let $X = a^i \frac{\partial}{\partial x^i}$ and $Y = b^j \frac{\partial}{\partial x^j}$ be two vector fields and let

$$g(X, Y) = \delta_{ij} a^i b^j. \tag{2.12}$$

The quantity $g(X, Y)$ is an example of a bilinear map which the reader will recognize as the usual dot product.

2.6 Definition A bilinear map $g(X, Y)$ on the tangent space is called a vector **inner product** if

1. $g(X, Y) = g(Y, X)$,
2. $g(X, X) \geq 0, \quad \forall X$,
3. $g(X, X) = 0$ iff $X = 0$.

Since we assume $g(X, Y)$ to be bilinear, an inner product is completely specified by its action on ordered pairs of basis vectors. The components g_{ij} of the inner product as thus given by

$$g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = g_{ij} \tag{2.13}$$

where g_{ij} is a symmetric $n \times n$ matrix, which we assume to be non-singular. By linearity, it is easy to see that if $X = a^i \frac{\partial}{\partial x^i}$ and $Y = b^j \frac{\partial}{\partial x^j}$ are two arbitrary vectors, then

$$g(X, Y) = g_{ij} a^i b^j.$$

In this sense, an inner product can be viewed as a generalization of the dot product. The standard Euclidean inner product is obtained if we take $g_{ij} = \delta_{ij}$. In this case the quantity $g(X, X) =\| X \|^2$ gives the square of the length of the vector. For this reason g_{ij} is also called a **metric** and g is called a **metric tensor**.

Another interpretation of the dot product can be seen if instead one considers a vector $X = a^i \frac{\partial}{\partial x^i}$ and a 1-form $\alpha = b_j dx^j$. The action of the 1-form on the vector gives

$$\begin{aligned}
\alpha(X) &= (b_j dx^j)(a^i \frac{\partial}{\partial x^i}) \\
&= b_j a^i (dx^j)(\frac{\partial}{\partial x^i}) \\
&= b_j a^i \delta^j_i \\
&= a^i b_i.
\end{aligned}$$

If we now define

$$b_i = g_{ij} b^j, \tag{2.14}$$

we see that the equation above can be rewritten as

$$a^i b_j = g_{ij} a^i b^j,$$

and we recover the expression for the inner product.

Equation (2.14) shows that the metric can be used as mechanism to lower indices, thus transforming the contravariant components of a vector to covariant ones. If we let g^{ij} be the inverse of the matrix g_{ij}, that is

$$g^{ik} g_{kj} = \delta^i_j, \tag{2.15}$$

we can also raise covariant indices by the equation

$$b^i = g^{ij} b_j \tag{2.16}$$

We have mentioned that the tangent and cotangent spaces of Euclidean space at a particular point are isomorphic. In view of the above discussion, we see that the metric accepts a dual interpretation; one as bilinear pairing of two vectors

$$g : T(\mathbf{R}^n) \times T(\mathbf{R}^n) \longrightarrow \mathbf{R}$$

and another as a linear isomorphism

$$g : T^\star(\mathbf{R}^n) \longrightarrow T(\mathbf{R}^n)$$

that maps vectors to covectors and vice-versa.

In elementary treatments of calculus authors often ignore the subtleties of differential 1-forms and tensor products and define the differential of arclength as

$$ds^2 \equiv g_{ij} dx^i dx^j,$$

although, what is really meant by such an expression is

$$ds^2 \equiv g_{ij} dx^i \otimes dx^j. \tag{2.17}$$

2.7 Example In cylindrical coordinates, the differential of arclength is

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2. \tag{2.18}$$

In this case the metric tensor has components

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.19}$$

2.8 Example In spherical coordinates

$$\begin{aligned} x &= \rho \sin\theta \cos\phi \\ y &= \rho \sin\theta \sin\phi \\ z &= \rho \cos\theta, \end{aligned} \tag{2.20}$$

the differential of arclength is given by

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2. \tag{2.21}$$

In this case the metric tensor has components

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin\theta^2 \end{bmatrix}. \tag{2.22}$$

Minkowski Space

An important object in mathematical physics is the so called Minkowski space which is can be defined as the pair Let $(\mathcal{M}_{1,3}, g)$ be the pair, where

$$\mathcal{M}_{(1,3)} = \{(t, x^1, x^2, x^3) |\ t, x^i \in \mathbf{R}\} \tag{2.23}$$

and g is the bilinear map such that

$$g(X, X) = -t^2 + (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{2.24}$$

The matrix representing Minkowski's metric g is given by

$$g = \text{diag}(-1, 1, 1, 1),$$

in which case, the differential of arclength is given by

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu \otimes dx^\nu \\ &= -dt \otimes dt + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \\ &= -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \end{aligned} \tag{2.25}$$

Note: Technically speaking, Minkowski's metric is not really a metric since $g(X, X) = 0$ does not imply that $X = 0$. Non-zero vectors with zero length are called Light-like vectors and they are associated with with particles which travel at the speed of light (which we have set equal to 1 in our system of units.)

The Minkowski metric $g_{\mu\nu}$ and its matrix inverse $g^{\mu\nu}$ are also used to raise and lower indices in the space in a manner completely analogous to $\mathbf{R}^n$. Thus, for example, if A is a covariant vector with components

$$A_\mu = (\rho, A_1, A_2, A_3),$$

then the contravariant components of A are

$$\begin{aligned} A^\mu &= g^{\mu\nu} A_\nu \\ &= (-\rho, A_1, A_2, A_3) \end{aligned}$$

Wedge Products and n-Forms

2.9 Definition A map $\phi : T(\mathbf{R}^n) \times T(\mathbf{R}^n) \longrightarrow \mathbf{R}$ is called **alternating** if

$$\phi(X, Y) = -\phi(Y, X)$$

The alternating property is reminiscent of determinants of square matrices which change sign if any two column vectors are switched. In fact, the determinant function is a perfect example of an alternating bilinear map on the space $M_{2\times 2}$ of two by two matrices. Of course, for the definition above to apply, one has to view $M_{2\times 2}$ as the space of column vectors.

2.10 Definition A **2-form** ϕ is a map $\phi : T(\mathbf{R}^n) \times T(\mathbf{R}^n) \longrightarrow \mathbf{R}$ which is alternating and bilinear.

2.11 Definition Let α and β be 1-forms in $\mathbf{R}^n$ and let X and Y be any two vector fields. The **wedge product** of the two 1-forms is the map $\alpha \wedge \beta : T(\mathbf{R}^n) \times T(\mathbf{R}^n) \longrightarrow \mathbf{R}$ given by the equation

$$(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X) \tag{2.26}$$

2.12 Theorem If α and β are 1-forms, then $\alpha \wedge \beta$ is a 2-form.
Proof: : We break up the proof into the following two lemmas.

2.13 Lemma The wedge product of two 1-forms is alternating.
Proof: Let α and β be 1-forms in $\mathbf{R}^n$ and let X and Y be any two vector fields. then

$$\begin{aligned}(\alpha \wedge \beta)(X,Y) &= \alpha(X)\beta(Y) - \alpha(Y)\beta(X) \\ &= -(\alpha(Y)\beta(X) - \alpha(X)\beta(Y)) \\ &= -(\alpha \wedge \beta)(Y,X)\end{aligned}$$

2.14 Lemma The wedge product of two 1-forms is bilinear.
Proof: Consider 1-forms, α, β, vector fields X_1, X_2, Y and functions f^1, F^2. Then, since the 1-forms are linear functionals, we get

$$\begin{aligned}(\alpha \wedge \beta)(f^1X_1 + f^2X_2, Y) &= \alpha(f^1X_1 + f^2X_2)\beta(Y) - \alpha(Y)\beta(f^1X_1 + f^2X_2) \\ &= [f^1\alpha(X_1) + f^2\alpha(X_2)]\beta(Y) - \alpha(Y)[f^1\beta(X_1) + f^2\alpha(X_2)] \\ &= f^1\alpha(X_1)\beta(Y) + f^2\alpha(X_2)\beta(Y) + f^1\alpha(Y)\beta(X_1) + f^2\alpha(Y)\beta(X_2) \\ &= f^1[\alpha(X_1)\beta(Y) + \alpha(Y)\beta(X_1)] + f^2[\alpha(X_2)\beta(Y) + \alpha(Y)\beta(X_2)] \\ &= f^1(\alpha \wedge \beta)(X_1, Y) + f^2(\alpha \wedge \beta)(X_2, Y)\end{aligned}$$

The proof of linearity on the second slot is quite similar and it is left to the reader.

2.15 Corollary If α and β are 1-forms, then

$$\alpha \wedge \beta = -\beta \wedge \alpha \tag{2.27}$$

This last result tells us that wedge products have characteristics similar to cross products of vectors in the sense that both of these products are anti-commutative. This means that we need to be careful to introduce a minus sign every time we interchange the order of the operation. Thus, for example, we have

$$dx^i \wedge dx^j = -dx^j \wedge dx^i$$

if $i \neq j$, whereas

$$dx^i \wedge dx^i = -dx^i \wedge dx^i = 0$$

since any quantity which is equal to the negative of itself must vanish. The similarity between wedge products is even more striking in the next proposition but we emphasize again that wedge products are by far much more powerful than cross products, because wedge products can be computed in any dimension.

2.16 Proposition Let $\alpha = A_i dx^i$ and $\beta = B_i dx^i$ be any two 1-forms in $\mathbf{R}^n$. Then

$$\alpha \wedge \beta = (A_i B_j) dx^i \wedge dx^j \tag{2.28}$$

Proof: Let X and Y be arbitrary vector fields. Then

$$\begin{aligned}(\alpha \wedge \beta)((X,Y) &= (A_i dx^i)(X)(B_j dx^j)(Y) - (A_i dx^i)(Y)(B_j dx^j)(X) \\ &= (A_i B_j)[dx^i(X)dx^j(Y) - dx^i(Y)dx^j(X)] \\ &= (A_i B_j)(dx^i \wedge dx^j)(X,Y)\end{aligned}$$

Because of the antisymmetry of the wedge product the last equation above can also be written as

$$\alpha \wedge \beta = \sum_{i=1}^{n}\sum_{j<i}^{n}(A_iB_j - A_jB_i)(dx^i \wedge dx^j)$$

In particular, if $n = 3$, then the coefficients of the wedge product are the components of the cross product of $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$

2.17 Example Let $\alpha = x^2dx - y^2dy$ and $\beta = dx + dy - 2xydz$. Then

$$\begin{aligned}
\alpha \wedge \beta &= (x^2dx - y^2dy) \wedge (dx + dy - 2xydz) \\
&= x^2dx \wedge dx + x^2dx \wedge dy - 2x^3ydx \wedge dz - y^2dy \wedge dx - y^2dy \wedge dy + 2xy^3dy \wedge dz \\
&= x^2dx \wedge dy - 2x^3ydx \wedge dz - y^2dy \wedge dx + 2xy^3dy \wedge dz \\
&= (x^2 + y^2)dx \wedge dy - 2x^3ydx \wedge dx + 2xy^3dy \wedge dz
\end{aligned}$$

2.18 Example let $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\begin{aligned}
dx \wedge dy &= (-r\sin\theta d\theta + \cos\theta dr) \wedge (r\cos\theta d\theta + \sin\theta dr) \\
&= -r\sin^2\theta d\theta \wedge dr + r\cos^2\theta dr \wedge d\theta \\
&= (r\cos^2\theta + r\sin^2\theta)(dr \wedge d\theta) \\
&= r(dr \wedge d\theta) \qquad (2.29)
\end{aligned}$$

2.19 Remark

1. The result of the last example yields the familiar differential of area in polar coordinates
2. The differential of area in polar coordinates is a special example of the change of coordinate theorem for multiple integrals. It is easy to establish that if $x = f^1(u, v)$ and $y = f^2(u, v)$, then $dx \wedge dy = det|J| du \wedge dv$, where $det|J|$ is the determinant of the Jacobian of the transformation.
3. Quantities such as $dxdy$ and $dydz$ which often appear in calculus, are not well defined. In most cases what is meant by these entities are wedge products of 1-forms
4. We state (without proof) that all 2-forms ϕ in $\mathbf{R}^n$ can be expressed as linear combinations of wedge products of differentials such as

$$\phi = F_{ij}dx^i \wedge dx^j \qquad (2.30)$$

In a more elementary (ie: sloppier) treatment of this subject one could simply define 2-forms to be gadgets which look like the quantity in equation (2.30). This is fact what we will do in the next definition.

2.20 Definition A 3=form ϕ in $\mathbf{R}^n$ is an object of the following type

$$\phi = A_{ijk}dx^i \wedge dx^j \wedge dx^k \qquad (2.31)$$

where we assume that the wedge product of three 1-forms is associative, but still alternating in the sense that if one switches any two differentials, then the entire expression changes by a minus sign. we challenge the reader to come up with a rigorous definition of three forms (or an n-form, for that matter) more in the spirit of multilinear maps. There is nothing really wrong with using definition ref3form. It is just that this definition is coordinate dependent and mathematicians in general (specially differential geometers) prefer coordinate-free definitions, theorems and proofs.

And now, a little combinatorics. Let us count the number of differential forms in Euclidean space. More specifically, we want to count the dimensions of the space of k-forms in $\mathbf{R}^n$ in the sense of vector spaces. We will thing of 0-forms as being ordinary functions. Since functions are the "scalars", the space of 0-forms as a vector space has dimension 1.

$\mathbf{R}^2$	Forms	Dim
0-forms	f	1
1-forms	fdx^1, gdx^2	2
2-forms	$fdx^1 \wedge dx^2$	1

$\mathbf{R}^3$	Forms	Dim
0-forms	f	1
1-forms	$f_1dx^1, f_2dx^2, f_3dx^3$	3
2-forms	$f_1dx^2 \wedge dx^3,\ f_2dx^3 \wedge dx^1,\ f_3dx^1 \wedge dx^2$	3
3-forms	$f_1dx^1 \wedge dx^2 \wedge dx^3$	1

The binomial coefficient pattern should be evident to the reader.

2.3 Exterior Derivatives

In this section we introduce a differential operator which generalizes the classical gradient, curl and divergence operators.

Denote by $\bigwedge^m_{(p)}(\mathbf{R}^n)$ the space of m-forms at $\mathbf{p} \in \mathbf{R}^n$. This vector space has dimension

$$\dim \bigwedge^m_{(p)}(\mathbf{R}^n) = \frac{n!}{m!(n-m)!}$$

for $m \leq n$ and dimension 0 if $m > n$. We shall identify $\bigwedge^0_{(p)}(\mathbf{R}^n)$ with the space of $\mathcal{C}^\infty$ functions at $\mathbf{p}$. Also we will call $\bigwedge^m(\mathbf{R}^n)$ the union of all $\bigwedge^m_{(p)}(\mathbf{R}^n)$ as $\mathbf{p}$ ranges through all the points in $\mathbf{R}^n$. In other words, we have

$$\bigwedge^m(\mathbf{R}^n) = \bigcup_p \bigwedge^m_p(\mathbf{R}^n).$$

If $\alpha \in \bigwedge^m(\mathbf{R}^n)$, then α can be written in the form

$$\alpha = A_{i_1,\ldots i_m}(x)dx^{i_1} \wedge \ldots dx^{i_m} \tag{2.32}$$

.

2.21 Definition Let α be an m-form (written in coordinates as in equation (2.32)). The **exterior derivative** of α is the (m+1-form) $d\alpha$ given by

$$\begin{aligned} d\alpha &= dA_{i_1,\ldots i_m} \wedge dx^{i_0} \wedge dx^{i_1} \ldots dx^{i_m} \\ &= \frac{\partial A_{i_1,\ldots i_m}}{\partial dx^{i_0}}(x)dx^{i_0} \wedge dx^{i_1} \ldots dx^{i_m} \end{aligned} \tag{2.33}$$

In the special case where α is a 0-form, that is, a function, we write

$$df = \frac{\partial f}{\partial x^i}dx^i$$

2.22 Proposition

$$
\begin{array}{ll}
\text{a)} & d : \bigwedge^m \longrightarrow \bigwedge^{m+1} \\
\text{b)} & d^2 = d \circ d = 0 \\
\text{c)} & d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \qquad \forall \alpha \in \bigwedge^p, \beta \in \bigwedge^q
\end{array}
\tag{2.34}
$$

Proof:

a) Obvious from equation (2.32).

b) First we prove the proposition for $\alpha = f \in \bigwedge^0$. We have

$$
\begin{aligned}
d(d\alpha) &= d(\frac{\partial f}{\partial dx^i}) \\
&= \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \\
&= \frac{1}{2}[\frac{\partial^2 f}{\partial x^j \partial x^i} \frac{\partial^2 f}{\partial x^i \partial x^j}] dx^j \wedge dx^i \\
&= 0
\end{aligned}
$$

Now, suppose that α is represented locally as in equation (2.32). It follows from 2.33 that

$$d(d\alpha) = d(dA_{i_1,\ldots i_m}) \wedge dx^{i_0} \wedge dx^{i_1} \ldots dx^{i_m} = 0$$

c) Let $\alpha \in \bigwedge^p, \beta \in \bigwedge^q$. Then we can write

$$
\begin{aligned}
\alpha &= A_{i_1,\ldots i_p}(x) dx^{i_1} \wedge \ldots dx^{i_p} \\
\beta &= B_{j_1,\ldots j_q}(x) dx^{j_1} \wedge \ldots dx^{j_q}.
\end{aligned}
\tag{2.35}
$$

By definition,

$$\alpha \wedge \beta = A_{i_1 \ldots i_p} B_{j_1 \ldots j_q} (dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \wedge (dx^{j_1} \wedge \ldots \wedge dx^{j_q})$$

Now we take take the exterior derivative of the last equation taking into account that $d(fg) = fdg + gdf$ for any functions f and g. We get

$$
\begin{aligned}
d(\alpha \wedge \beta) &= [d(A_{i_1 \ldots i_p}) B_{j_1 \ldots j_q} + (A_{i_1 \ldots i_p} d(B_{j_1 \ldots j_q})](dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \wedge (dx^{j_1} \wedge \ldots \wedge dx^{j_q}) \\
&= [dA_{i_1 \ldots i_p} \wedge (dx^{i_1} \wedge \ldots \wedge dx^{i_p})] \wedge [B_{j_1 \ldots j_q} \wedge (dx^{j_1} \wedge \ldots \wedge dx^{j_q})] + \\
&= [A_{i_1 \ldots i_p} \wedge (dx^{i_1} \wedge \ldots \wedge dx^{i_p})] \wedge (-1)^p [dB_{j_1 \ldots j_q} \wedge (dx^{j_1} \wedge \ldots \wedge dx^{j_q})] \\
&= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.
\end{aligned}
\tag{2.36}
$$

The $(-1)^p$ factor comes in because to pass the term $dB_{j_i \ldots j_p}$ through p 1-forms of the type dx^i, one has to perform p transpositions.

2.23 Example Let $\alpha = P(x,y)dx + Q(x,y)d\beta$. Then,

$$
\begin{aligned}
d\alpha &= (\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y}) \wedge dx + (\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y}) \wedge dy \\
&= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy \\
&= (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx \wedge dy.
\end{aligned}
\tag{2.37}
$$

This example is related to Green's theorem in $\mathbf{R}^2$.

2.24 Example Let $\alpha = M(x,y)dx + N(x,y)dy$, and suppose that $d\alpha = 0$. Then, by the previous example,

$$d\alpha = (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dx \wedge dy.$$

Thus, $d\alpha = 0$ iff $N_x = M_y$ which implies that $N = f_y$ and M_x for some $\mathcal{C}^1$ function $f(x,y)$. Hence

$$\alpha = f_x dx + f_y df = df.$$

The reader should also be familiar with this example in the context of exact differential equations of first order, and conservative force fields.

2.25 Definition A differential form α is called **exact** if $d\alpha = 0$.

2.26 Definition A differential form α is called **closed** if there exist a form β such that $\alpha = d\beta$. Since $d \circ d = 0$, it is clear that a closed form is also exact. The converse is not at all obvious in general and we state it here without proof.

2.27 Poincare's Lemma In a simply connected space (such as $\mathbf{R}^n$), if a differential is exact then it is closed.
The assumption hypothesis that the space must be simply connected is somewhat subtle. The condition is reminiscent of Cauchy's integral theorem for functions of a complex variable, which states that if $f(z)$ is holomorphic function and C is a simple closed curve, then,

$$\oint_C f(z)dz = 0$$

This theorem does not hold if the region bounded by the curve C is not simply connected. The standard example is the integral of the complex 1-form $\omega = (1/z)dz$ around the unit circle C bounding a punctured disk. In this case,

$$\oint_C \frac{1}{z}dz = 2\pi i$$

2.4 The Hodge-$*$ Operator

One of the important lessons that students learn in linear algebra is that all vector space of finite dimension n are isomorphic to each other. Thus, for instance, the space P_3 of all real polynomials in x of degree 3, and the space $\mathcal{M}_{2\times 2}$ of real 2 by 2 matrices, are basically no different than the Euclidean vector space $\mathbf{R}^4$ in terms of their vector space properties. We have already encountered a number of vector spaces of finite dimension in these notes. A good example of this is the tangent space $T_p\mathbf{R}^3$. The "vector" part $a^1\frac{\partial}{\partial x} + a^2\frac{\partial}{\partial y} + a^3\frac{\partial}{\partial z}$ can be mapped to a regular advanced calculus vector $a^1\mathbf{i} + a^2\mathbf{j} + a^3\mathbf{k}$, by replacing $\frac{\partial}{\partial x}$ by $\mathbf{i}$, $\frac{\partial}{\partial y}$ by $\mathbf{j}$ and $\frac{\partial}{\partial z}$ by $\mathbf{k}$. Of course, we must not confuse a tangent vector which is a linear operator with a Euclidean vector which is just an ordered triplet, but as far their vector space properties, there is basically no difference.

We have also observed that the tangent space $T_p\mathbf{R}^n$ is isomorphic to the cotangent space $T_p^\star\mathbf{R}^n$. In this case, the vector space isomorphism maps the standard basis vectors $\{\frac{\partial}{\partial x^i}\}$ to their duals $\{dx^i\}$. This isomorphism then transforms a contravariant vector to a covariant vector.

Another interesting example is provided by the spaces $\bigwedge_p^1(\mathbf{R}^3)$ and $\bigwedge_p^2(\mathbf{R}^3)$, both of which have dimension 3. It follows that these two spaces must be isomorphic. In this case the isomorphism is given by the map

$$dx \quad \longmapsto \quad dy \wedge dz$$

$$\begin{aligned} dy &\longmapsto -dx \wedge dz \\ dz &\longmapsto dx \wedge dy \end{aligned} \tag{2.38}$$

More generally, we have seen that the dimension of the space of m-forms in $\mathbf{R}^n$ is given by the binomial coefficient $\binom{n}{m}$. Since

$$\binom{n}{m} = \binom{n}{n-m} = \frac{n!}{(n-m)!},$$

it must be the case that

$$\textstyle\bigwedge_p^m(\mathbf{R}^n) \cong \bigwedge_p^m(\mathbf{R}^{n-m}) \tag{2.39}$$

To describe the isomorphism between these two spaces, we will first need to introduce the totally antisymmetric **Levi-Civita** permutation symbol which is defined as follows

$$\epsilon_{i_1 \ldots i_m} = \begin{cases} +1 & \text{if}(i_1, \ldots, i_m) \text{ is an even permutation of}(1, \ldots, m) \\ -1 & \text{if}(i_1, \ldots, i_m) \text{ is an odd permutation of}(1, \ldots, m) \\ 0 & \text{otherwise} \end{cases} \tag{2.40}$$

In dimension 3, there are only 3 (3!=6) nonvanishing components of $\epsilon_{i,j,k}$ in

$$\begin{aligned} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = 1 \end{aligned} \tag{2.41}$$

The permutation symbols are useful in the theory of determinants. In fact, if $A = (a^i_j)$ is a 3×3 matrix, then, using equation (2.41), the reader can easily verify that

$$\det A = |A| = \epsilon_{i_1 i_2 i_3} a_1^{i_1} a_2^{i_2} a_3^{i_3} \tag{2.42}$$

This formula for determinants extends in an obvious manner to $n \times n$ matrices. A more thorough discussion of the Levi-Civita symbols will appear later in these notes.

In $\mathbf{R}^n$, the Levi-Civita symbol with some or all the indices up is numerically equal to the permutation symbol will all indices down

$$\epsilon_{i_1 \ldots i_m} = \epsilon^{i_1 \ldots i_m},$$

since the Euclidean metric used to raise and lower indices is δ_{ij}.

On the other hand, in Minkowski space, raising an index with a value of 0 costs a minus sign, because $g_{00} = g^{00} = -1$. Thus, in $calM_{(1,3)}$

$$\epsilon_{i_0 i_1 i_2 i_3} = -\epsilon^{i_0 i_1 i_2 i_3},$$

since any permutation of $\{0, 1, 2, 3\}$ must contain a 0.

2.28 Definition The Hodge-$*$ operator is a linear map $* : \bigwedge_p^m(\mathbf{R}^n) \longrightarrow \bigwedge_p^m(\mathbf{R}^{n-m})$ defined in standard local coordinates by the equation

$$*(dx^{i_1} \wedge \ldots \wedge dx^{i_m}) = \frac{1}{(n-m)!} \epsilon^{i_1 \ldots i_m}{}_{i_{m+1} \ldots i_n} dx^{i_{m+1}} \wedge \ldots \wedge dx^{i_m}, \tag{2.43}$$

Since the forms $dx^{i_1} \wedge \ldots \wedge dx^{i_m}$ constitute a basis of the vector space $\bigwedge_p^m(\mathbf{R}^n)$ and the $*$-operator is assumed to be a linear map, equation (2.43) completely specifies the map for all m-forms.

2.29 Example Consider the dimension n=3 case. then

$$\begin{aligned} *dx^1 &= \epsilon^1{}_{jk}dx^j \wedge dx^k \\ &= \frac{1}{2!}[\epsilon^1{}_{23}dx^2 \wedge dx^3 + \epsilon^1{}_{32}dx^3 \wedge dx^2] \\ &= \frac{1}{2!}[dx^2 \wedge dx^3 - dx^3 \wedge dx^2] \\ &= \frac{1}{2!}[dx^2 \wedge dx^3 + dx^2 \wedge dx^3] \\ &= dx^2 \wedge dx^3. \end{aligned}$$

We leave it to reader to complete the computation of the action of the $*$-operator on the other basis forms. The results are

$$\begin{aligned} *dx^1 &= +dx^2 \wedge dx^3 \\ *dx^2 &= -dx^1 \wedge dx^3 \\ *dx^3 &= +dx^1 \wedge dx^2, \end{aligned} \tag{2.44}$$

$$\begin{aligned} *(dx^2 \wedge dx^3) &= dx^1 \\ *(-dx^3 \wedge dx^1) &= dx^2 \\ *(dx^1 \wedge dx^2) &= dx^3, \end{aligned} \tag{2.45}$$

and

$$*(dx^1 \wedge dx^2 \wedge dx^3) = 1. \tag{2.46}$$

In particular, if $f : \mathbf{R}^3 \longrightarrow \mathbf{R}$ is any 0-form (a function) then,

$$\begin{aligned} *f &= f(dx^1 \wedge dx^2 \wedge dx^3) \\ &= fdV, \end{aligned} \tag{2.47}$$

where dV is the differential of volume, also called the volume form.

2.30 Example Let $\alpha = A_1dx^1A_2dx^2 + A_3dx^3$, and $\beta = B_1dx^1B_2dx^2 + B_3dx^3$. Then,

$$\begin{aligned} *(\alpha \wedge \beta) &= (A_2B_3 - A_3B_2) * (dx^2 \wedge dx^3) + (A_1B_3 - A_3B_1) * (dx^1 \wedge dx^3) + \\ &\quad (A_1B_2 - A_2B_1) * (dx^1 \wedge dx^2) \\ &= (A_2B_3 - A_3B_2)dx^1 + (A_1B_3 - A_3B_1)dx^2 + (A_1B_2 - A_2B_1)dx^3 \\ &= (\vec{\mathbf{A}} \times \vec{\mathbf{B}})_i dx^i \end{aligned} \tag{2.48}$$

The previous examples provide some insight on the action of the $\wedge$ and $*$ operators. If one thinks of the quantities dx^1, dx^2 and dx^3 as playing the role of $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$ and $\vec{\mathbf{k}}$, then it should be apparent that equations 2.44 are the differential geometry versions of the well known relations

$$\begin{aligned} \mathbf{i} &= \mathbf{j} \times \mathbf{k} \\ \mathbf{j} &= -\mathbf{i} \times \mathbf{k} \\ \mathbf{k} &= \mathbf{i} \times \mathbf{j} \end{aligned}$$

. This is even more evident upon inspection of equation (2.48), which relates the $\wedge$ operator to the Cartesian cross product.

2.31 Example In Minkowski space the collection of all 2-forms has dimension $\binom{4}{2} = 6$. The Hodge-$*$ operator in this case, splits $\bigwedge^2(\mathcal{M}_{1,3})$ into two 3-dim subspaces $\bigwedge^2_{\pm}$, such that $* : \bigwedge^2_{\pm} \longrightarrow \bigwedge^2_{\mp}$. More specifically, $\bigwedge^2_{+}$ is spanned by the forms $\{dx^0 \wedge dx^1, dx^0 \wedge dx^2, dx^0 \wedge dx^3\}$, and $\bigwedge^2_{-}$ is spanned by the forms $\{dx^2 \wedge dx^3, -dx^1 \wedge dx^3, dx^1 \wedge dx^2\}$. The action of $*$ on $\bigwedge^2_{+}$ is

$$
\begin{aligned}
*(dx^0 \wedge dx^1) = &\ \tfrac{1}{2}\epsilon^{01}{}_{kl}dx^k \wedge dx^l = & -dx^2 \wedge dx^3 \\
*(dx^0 \wedge dx^2) = &\ \tfrac{1}{2}\epsilon^{02}{}_{kl}dx^k \wedge dx^l = & +dx^1 \wedge dx^3 \\
*(dx^0 \wedge dx^3) = &\ \tfrac{1}{2}\epsilon^{03}{}_{kl}dx^k \wedge dx^l = & -dx^1 \wedge dx^2,
\end{aligned}
$$

and on $\bigwedge^2_{-}$,

$$
\begin{aligned}
*(+dx^2 \wedge dx^3) = &\ \tfrac{1}{2}\epsilon^{23}{}_{kl}dx^k \wedge dx^l = & dx^0 \wedge dx^1 \\
*(-dx^1 \wedge dx^3) = &\ \tfrac{1}{2}\epsilon^{13}{}_{kl}dx^k \wedge dx^l = & dx^0 \wedge dx^2 \\
*(+dx^1 \wedge dx^2) = &\ \tfrac{1}{2}\epsilon^{12}{}_{kl}dx^k \wedge dx^l = & dx^0 \wedge dx^3,
\end{aligned}
$$

In verifying the equations above, we recall that the Levi-Civita symbols which contain an index with value 0 in the up position have an extra minus sign as a result of raising the index with g^{00}.If $F \in \bigwedge^2(\mathcal{M})$, we will formally write $F = F_+ + F_-$, where $F_{\pm} \in \bigwedge^2_{\pm}$. We would like to note that the action of the dual operator on $\bigwedge^2(\mathcal{M})$ is such that $* \bigwedge^2(\mathcal{M}) \longrightarrow \bigwedge^2(\mathcal{M})$, and $*^2 = -1$. Thus, the operator is an linear involution of the space and in fact, $\bigwedge^2_{\pm}$ are the eigenspaces corresponding to the two eigenvalues of this involution.

It is also worthwhile to calculate the duals of 1-forms in $\mathcal{M}_{1,3}$. The results are

$$
\begin{aligned}
*dt &= -dx^1 \wedge dx^2 \wedge dx^3 \\
*dx^1 &= +dx^2 \wedge dt \wedge dx^3 \\
*dx^2 &= +dt \wedge dx^1 \wedge dx^3 \\
*dx^3 &= +dx^1 \wedge dt \wedge dx^2.
\end{aligned} \tag{2.49}
$$

Gradient, Curl and Divergence

Classical differential operators which enter in Green's and Stokes Theorems are better understood as special manifestations of the exterior differential and the Hodge-$*$ operators in $\mathbf{R}^3$. Here is precisely how this works:

1. Let $f : \mathbf{R}^3 \longrightarrow \mathbf{R}$ be a $\mathcal{C}^\infty$ function. Then

$$df = \frac{\partial f}{\partial x^j} dx^j = \nabla f \cdot \mathbf{dx} \tag{2.50}$$

2. Let $\alpha = A_i dx^i$ be a 1-form in $\mathbf{R}^3$. Then

$$
\begin{aligned}
(*d)\alpha &= \frac{1}{2}\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_i}{\partial x^j}\right) * (dx^i \wedge dx^j) \\
&= (\nabla \times \mathbf{A}) \cdot \mathbf{dx}
\end{aligned} \tag{2.51}
$$

3. Let $\alpha = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2$ be a 2-form in $\mathbf{R}^3$. Then

$$
\begin{aligned}
d\alpha &= \left(\frac{\partial B_1}{\partial x^1} + \frac{\partial B_2}{\partial x^2} + \frac{\partial B_3}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3 \\
&= (\nabla \cdot \mathbf{B}) dV
\end{aligned} \tag{2.52}
$$

4. Let $\alpha = B_i dx^i$, then
$$*d*\alpha = \nabla \cdot \mathbf{B} \tag{2.53}$$

It is also possible to define and manipulate formulas of classical vector calculus using the permutation symbols. For example, let $\mathbf{a} = (A_1, A_2, A_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$ be any two Euclidean vectors. Then it is easy to see that

$$(\mathbf{A} \times \mathbf{B})_k = \epsilon^{ij}{}_k A_i B_j,$$

and

$$(\nabla \times \mathbf{B})_k = \epsilon^{ij}{}_k \frac{\partial A_i}{\partial x^j},$$

To derive many classical vector identities in this formalism, it is necessary to first establish the following identity (see Ex. ())
$$\epsilon^{ijm}\epsilon_{klm} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k \tag{2.54}$$

2.32 Example

$$\begin{aligned}
[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_l &= \epsilon_l^{mn} A\ {}_m (\mathbf{B} \times \mathbf{C})_m \\
&= \epsilon_l^{mn} A_m (\epsilon^{jk}_{\ n} B_j C_k) \\
&= \epsilon^{mn}_{\ l} \epsilon^{jk}_{\ n} A_m B_j C_k) \\
&= \epsilon_{mnl} \epsilon^{jkn} A^m B_j C_k) \\
&= (\delta^j_l \delta^k_m - \delta^k_l \delta^j_n) A^m B_j C_k \\
&= B_l A^m C_m - C_l A^m b_n
\end{aligned}$$

Or, rewriting in vector form

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{2.55}$$

Maxwell Equations

The classical equations of Maxwell describing electromagnetic phenomena are

$$\begin{array}{ll}
\nabla \cdot \mathbf{E} = 4\pi\rho & \nabla \times \mathbf{B} = 4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \\
\nabla \cdot \mathbf{B} = 0 & \nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}
\end{array} \tag{2.56}$$

We would like to formulate these equations in the language of differential forms. Let $x^\mu = (t, x^1, x^2, x^3)$ be local coordinates in Minkowski's space $\mathcal{M}_{1,3}$. Define the Maxwell 2-form F by the equation

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad (\mu, \nu = 0, 1, 2, 3), \tag{2.57}$$

where

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_y \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}. \tag{2.58}$$

Written in complete detail, Maxwell's 2-form is given by

$$\begin{aligned}
F = & -E_x dt \wedge dx^1 - E_y dt \wedge dx^2 - E_z dt \wedge dx^3 + \\
& B_z dx^1 \wedge dx^2 - B_y dx^1 \wedge dx^3 + B_x dx^2 \wedge dx^3.
\end{aligned} \tag{2.59}$$

We also define the source current 1-form

$$J = J_\mu dx^\mu = \rho dt + J_1 dx^1 + J_2 dx^2 + J_3 dx^3. \tag{2.60}$$

2.33 Proposition Maxwell's Equations 2.56 are equivalent to the equations

$$\begin{aligned} dF &= 0, \\ d * F &= 4\pi * J. \end{aligned} \tag{2.61}$$

Proof: The proof is by direct computation using the definitions of the exterior derivative and the Hodge-$*$ operator.

$$\begin{aligned} dF = & -\frac{\partial E_x}{\partial x^2} \wedge dx^2 \wedge dt \wedge dx^1 - \frac{\partial E_x}{\partial x^3} \wedge dx^3 \wedge dt \wedge dx^1 + \\ & -\frac{\partial E_y}{\partial x^1} \wedge dx^1 \wedge dt \wedge dx^2 - \frac{\partial E_y}{\partial x^3} \wedge dx^3 \wedge dt \wedge dx^2 + \\ & -\frac{\partial E_z}{\partial x^1} \wedge dx^1 \wedge dt \wedge dx^3 - \frac{\partial E_z}{\partial x^2} \wedge dx^2 \wedge dt \wedge dx^3 + \\ & \frac{\partial B_z}{\partial t} \wedge dt \wedge dx^1 \wedge dx^2 - \frac{\partial B_z}{\partial x^3} \wedge dx^3 \wedge dx^1 \wedge dx^2 - \\ & \frac{\partial B_y}{\partial t} \wedge dt \wedge dx^1 \wedge dx^3 - \frac{\partial B_y}{\partial x^2} \wedge dx^2 \wedge dx^1 \wedge dx^3 + \\ & \frac{\partial B_x}{\partial t} \wedge dt \wedge dx^2 \wedge dx^3 + \frac{\partial B_x}{\partial x^1} \wedge dx^1 \wedge dx^2 \wedge dx^3. \end{aligned}$$

Collecting terms and using the antisymmetry of the wedge operator, we get,

$$\begin{aligned} dF = & (\frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3}) dx^1 \wedge dx^2 \wedge dx^3 + \\ & (\frac{\partial E_y}{\partial x^3} - \frac{\partial E_z}{\partial x^2} - \frac{\partial B_x}{\partial t}) dx^2 \wedge dt \wedge dx^3 + \\ & (\frac{\partial E_z}{\partial x^1} - \frac{\partial E_x}{\partial dx^3} - \frac{\partial B_y}{\partial t}) dt \wedge dx^1 \wedge x^3 + \\ & (\frac{\partial E_x}{\partial x^2} - \frac{\partial E_y}{\partial x^1} - \frac{\partial B_z}{\partial t}) dx^1 \wedge dt \wedge dx^2. \end{aligned}$$

Therefore, $dF = 0$ iff

$$\frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_y}{\partial x^3} = 0,$$

which is the same as

$$\nabla \cdot \mathbf{B} = 0,$$

and

$$\begin{aligned} \frac{\partial E_y}{\partial x^3} - \frac{\partial E_z}{\partial x^2} - \frac{\partial B_x}{\partial x^1} &= 0, \\ \frac{\partial E_z}{\partial x^1} - \frac{\partial E_x}{\partial x^3} - \frac{\partial B_y}{\partial x^2} &= 0, \\ \frac{\partial E_x}{\partial x^2} - \frac{\partial E_y}{\partial x^1} - \frac{\partial B_z}{\partial x^3} &= 0, \end{aligned}$$

which means that

$$-\nabla \times \mathbf{E} - \frac{\partial B}{\partial t} = 0. \tag{2.62}$$

To verify the second set of Maxwell equations, we first compute the dual of the current density 1-form (2.60) using the results from example 2.4. We get

$$*J = -\rho dx^1 \wedge dx^2 \wedge dx^3 + J_1 dx^2 \wedge dt \wedge dx^3 + J_2 dt \wedge dx^1 \wedge dx^3 + J_3 dx^1 \wedge dt \wedge dx^2. \tag{2.63}$$

We could now proceed to compute $d{*}F$, but perhaps it is more elegant to notice that $F \in \bigwedge^2(\mathcal{M})$, and so, according to example (2.4), F splits into $F = F_+ + F_-$. In fact, we see from (2.58) that the components of F_+ are those of $-\mathbf{E}$ and the components of F_- constitute the magnetic field vector $\mathbf{B}$. Using the results of example (2.4), we can immediately write the components of $*F$

$$\begin{aligned} *F &= B_x dt \wedge dx^1 + B_y dt \wedge dx^2 + B_z dt \wedge dx^3 + \\ & \quad E_z dx^1 \wedge dx^2 - E_y dx^1 \wedge dx^3 + E_x dx^2 \wedge dx^3, \end{aligned} \tag{2.64}$$

or equivalently,

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_y \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{bmatrix}. \tag{2.65}$$

Since the effect of the dual operator amounts to exchanging

$$\begin{aligned} \mathbf{E} &\longmapsto -\mathbf{B} \\ \mathbf{B} &\longmapsto +\mathbf{E}, \end{aligned}$$

we can infer from equations (2.62) and (2.63) that

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

and,

$$\nabla \times \mathbf{B} - \frac{\partial E}{\partial t} = 4\pi \mathbf{J}.$$